

Irreducibility of Infinite Dimensional Steinberg Modules of Reductive Groups with Frobenius Maps

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ABSTRACT. Let G be a connected reductive group over an algebraic closure $\bar{\mathbb{F}}_q$ of a finite field \mathbb{F}_q . In this paper it is proved that the infinite dimensional Steinberg module of kG defined by N. Xi in 2014 is irreducible when k is a field of positive characteristic and $\text{char } k \neq \text{char } \mathbb{F}_q$. For certain special linear groups, we show that the Steinberg modules of the groups are not quasi-finite with respect to some natural quasi-finite sequences of the groups.

N. Xi studied some induced representations of infinite reductive groups with Frobenius maps (see [X]). In particular, he defined Steinberg modules for any reductive groups by extending Steinberg's construction of Steinberg modules for finite reductive groups. These Steinberg modules are infinite dimensional when the reductive groups are infinite.

Let G be a connected reductive group over the algebraic closure $\bar{\mathbb{F}}_q$ of a finite field \mathbb{F}_q and k a field. Xi proved that the Steinberg module of the group algebra kG of G over k is irreducible if k is the field of complex numbers or $k = \bar{\mathbb{F}}_q$ (In fact, his proof works when $\text{char } k=0$ or $\text{char } \mathbb{F}_q$). In this paper we prove that if k has positive characteristic and $\text{char } k \neq \text{char } \mathbb{F}_q$, then the Steinberg module of kG remains irreducible (see Theorem 2.2).

The reductive group G is quasi-finite in the sense of [X, 1.8]. For quasi-finite groups Xi introduced the concept of quasi-finite irreducible module and raised the question whether an irreducible kG -module is

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always quasi-finite. For certain special linear groups, we show that the Steinberg modules of the groups are not quasi-finite with respect to some natural quasi-finite sequences of the groups (see Proposition 3.2).

1. Preliminaries

In this section we recall some basic facts for reductive groups defined over a finite field, for details we refer to [C].

1.1. Let G be a connected reductive group over an algebraically closure $\bar{\mathbb{F}}_q$ of a finite field \mathbb{F}_q of q elements, where q is a power of a prime p . Assume that G is defined over \mathbb{F}_q . Then G has a Borel subgroup B defined over \mathbb{F}_q and B contains a maximal torus T defined over \mathbb{F}_q . The unipotent radical U of B is defined over \mathbb{F}_q . For any power q^a of q , we denote by G_{q^a} the \mathbb{F}_{q^a} -points of G and shall identify G with its $\bar{\mathbb{F}}_q$ -points. Then we have $G = \bigcup_{a=1}^{\infty} G_{q^a}$. Similarly we define B_{q^a} , T_{q^a} and U_{q^a} .

1.2. Let $N = N_G(T)$ be the normalizer of T in G . Then B and N form a BN -pair of G . Let $R \subset \text{Hom}(T, \bar{\mathbb{F}}_q^*)$ be the root system of G and R^+ the set of positive roots determined by B . For $\alpha \in R^+$, let U_α be the corresponding root subgroup of U .

For any simple root α in R , let s_α be the corresponding simple reflection in the Weyl group $W = N/T$. For $w \in W$, U has two subgroups U_w and U'_w such that $U = U'_w U_w$ and $wU'_w w^{-1} \subseteq U$. If $w = s_\alpha$ for some simple root α , then $U_w = U_\alpha$ and we simply write U'_α for U'_w , which equals $\prod_{\beta \in R^+ - \{\alpha\}} U_\beta$. In general, let $w = s_{\alpha_i} \cdots s_{\alpha_2} s_{\alpha_1}$ be a reduced expression of w . Set $\beta_j = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{j-1}}(\alpha_j)$ for $j = 1, \dots, i$. Then

$$(a) \quad U_w = U_{\beta_i} \cdots U_{\beta_2} U_{\beta_1} \text{ and } U'_w = \prod_{\substack{\beta \in R^+ \\ w(\beta) \in R^+}} U_\beta.$$

(b) If α and β are positive roots and $w(\alpha) = \beta$, then $n_w U_\alpha n_w^{-1} = U_\beta$, where n_w is a representative of w in N .

Now assume that $w_0 = s_{\alpha_r} \cdots s_{\alpha_2} s_{\alpha_1}$ is a reduced expression of the longest element of W . Set $\beta_j = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{j-1}}(\alpha_j)$ for $j = 1, \dots, r$. Then

(c) For any $1 \leq i \leq j \leq r$, $U_{\beta_j} \cdots U_{\beta_{i+1}} U_{\beta_i}$ is a subgroup of U and $U_{\beta_j} \cdots U_{\beta_{i+1}} U_{\beta_i} = U_{\beta_i} U_{\beta_j} \cdots U_{\beta_{i+1}}$.

The roots subgroups U_α , $\alpha \in R^+$, are also defined over \mathbb{F}_q . For each positive root, we fix an isomorphism $\varepsilon_\alpha : \bar{\mathbb{F}}_q \rightarrow U_\alpha$ such that $t\varepsilon_\alpha(c)t^{-1} = \varepsilon_\alpha(\alpha(t)c)$. Set $U_{\alpha, q^a} = \varepsilon_\alpha(\mathbb{F}_{q^a})$.

2. Infinite dimensional Steinberg modules

In this section the main result (Theorem 2.2) of this paper is proved, which says that certain infinite dimensional Steinberg modules are irreducible.

2.1. Let k be a field. For any one dimensional representation θ of T over k , let k_θ be the corresponding kT -module. We define the kG -module $M(\theta) = kG \otimes_{kB} k_\theta$. When θ is trivial representation of T over k , we write $M(tr)$ for $M(\theta)$ and let 1_{tr} be a nonzero element in k_θ . We shall also write $x1_{tr}$ instead of $x \otimes 1_{tr}$ for $x \in kG$.

For $w \in W = N/T$, the element $w1_{tr}$ is defined to be $n_w 1_{tr}$, where n_w is a representative in N of w . This is well defined since T acts on k_θ trivially. Let $\eta = \sum_{w \in W} (-1)^{l(w)} w1_{tr} \in M(tr)$, where $l : W \rightarrow \mathbb{N}$ is the length function of W . Then $kU\eta$ is a submodule of $M(tr)$ and is called a Steinberg module of G , denoted by St , see [X, Prop. 2.3]. Xi proved that St is irreducible if k is the field of complex numbers or $k = \bar{\mathbb{F}}_q$ (see [X, Theorem 3.2]). His argument in fact works for proving that St is irreducible whenever $\text{char} k = 0$ or $\text{char} k = \text{char} \mathbb{F}_q$. The main result of this paper is the following.

2.2. Theorem. Assume that k is a field of positive characteristic and $\text{char} k \neq \text{char} \mathbb{F}_q$. Then the Steinberg module St is irreducible.

Combining Xi's result we have the following result.

2.3. Corollary. The Steinberg module St of kG is irreducible for any field k .

2.4. We need some preparation to prove the theorem.

Let $s_{\alpha_r} s_{\alpha_{r-1}} \cdots s_{\alpha_1}$ be a reduced expression of the longest element w_0 of W . Set $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$. Then R^+ consists of $\beta_r, \beta_{r-1}, \dots, \beta_1$, and $U = U_{\beta_r} U_{\beta_{r-1}} \cdots U_{\beta_1}$. Let n_i be a representative in $N = N_G(T)$ of s_{α_i} and set $n = n_r n_{r-1} \cdots n_1$. Note that the elements $z\eta$, $z \in U$, form a basis of St .

2.5. Lemma. Let $u \in U_{q^a}$. If u is not the neutral element e of U , then the sum of all coefficients of $nu\eta$ in terms the basis $z\eta$, $z \in U$, is 0.

Proof. Let α_i and β_i be as in subsection 2.4. Then $u = u_r u_{r-1} \cdots u_1$, $u_m \in U_{\beta_m}$. Assume that $u_1 = u_2 = \cdots u_{i-1} = e$ but $u_i \neq e$, where e is the neutral element of G . We use induction on i to prove the lemma. Note that $n_i \eta = -\eta$ for $i = 1, 2, \dots, r$.

Assume that $i = r$. Then $nu\eta = (-1)^{r-1} n_r u'_r \eta$, where $u'_r = n_{r-1} \cdots n_1 u_r n_1^{-1} \cdots n_{r-1}^{-1} \in U_{\alpha_r}$. According to the proof of [S, Lemma 1] (see also proof of [X, Proposition 2.3]), there exists $x_r \in U_{\alpha_r}$ such that $n_r u'_r \eta = (x_r - 1)\eta$. So the lemma is true in this case.

Now assume that the lemma is true for $r, r-1, \dots, i+1$, we show that it is also true for i . In this case we have $nu = (-1)^{i-1} n_r \cdots n_i u'_i u'_{i-1} \cdots u'_1 \eta$, where $u'_j = n_{i-1} \cdots n_1 u_j n_1^{-1} \cdots n_{i-1}^{-1} \in n_{i-1} \cdots n_1 U_{\beta_j} n_1^{-1} \cdots n_{i-1}^{-1} = U_{\gamma_j}$, where $\gamma_j = s_{\alpha_i} \cdots s_{\alpha_{j-1}}(\alpha_j)$, $j = i, i+1, \dots, r$. Then $u'_i \neq e$. Note that $\gamma_i = \alpha_i$. According to the proof of [S, Lemma 1], there exists $x_i \in U_{\gamma_i}$ such that $n_i u'_i \eta = (x_i - 1)\eta$.

If $u'_r \cdots u'_{i+1} = e$, we are done. Now assume that $u' = u'_r \cdots u'_{i+1} \neq e$. Since both $M_i = U_{\gamma_r} U_{\gamma_{r-1}} \cdots U_{\gamma_i}$ and $M_{i+1} = U_{\gamma_r} U_{\gamma_{r-1}} \cdots U_{\gamma_{i+1}}$ are subgroups of U and $M_{i+1} x_i = x_i M_{i+1}$, we see that $u' x_i = x_i u''$ for some $u'' \in M_{i+1}$ and $u'' \neq e$. Thus

$$(-1)^{i-1} nu\eta = n_r \cdots n_{i+1} u' (x_i - 1)\eta = n_r \cdots n_{i+1} x_i u'' \eta - n_r \cdots n_{i+1} u' \eta.$$

By induction hypotheses, we know that the sum of the coefficients of $n_r \cdots n_{i+1} u'' \eta$ and the sum of the coefficients of $n_r \cdots n_{i+1} u' \eta$ are 0. Since $n_r \cdots n_{i+1} x_i n_{i+1}^{-1} \cdots n_r^{-1} \in U$, we see that the sum of the coefficients of $nu\eta = (-1)^{i-1} n_r \cdots n_{i+1} u' (x_i - 1)\eta$ is 0.

The lemma is proved.

2.6. Lemma. Let V be a nonzero submodule of St . Then there exists an integer a such that $\sum_{x \in U_{q^a}} x\eta$ is in V .

Proof. Let v be a nonzero element in V . Then $v \in kU_{q^a}\eta$ for some integer a . Let $v = \sum_{y \in U_{q^a}} a_y y\eta$. We may assume that $a_e \neq 0$. Otherwise choose $y \in U_{q^a}$ such that a_y is nonzero and replace v by $y^{-1}v$.

By Lemma 2.5 we see that the sum A of all the coefficients of nv in terms of the basis $z\eta$, $z \in U$, is $(-1)^{l(w_0)}a_e \neq 0$. Thus $\sum_{x \in U_{q^a}} xnv = A \sum_{x \in U_{q^a}} x\eta$. The lemma is proved.

2.7. Now we can prove the theorem. We show that $\text{St} = kGv$ for any nonzero element v in St . Let $V = kGv$. Let α_i and β_i be as in subsection 2.4. For any positive integer b , set $X_{i,q^b} = U_{\beta_r,q^b}U_{\beta_{r-1},q^b} \cdots U_{\beta_i,q^b}$. Then X_{i,q^b} is a subgroup of U and $X_{i,q^b} = X_{i+1,q^b}U_{\beta_i,q^b}$. Clearly X_{i,q^b} is a subgroup $X_{i,q^{b'}}$ if \mathbb{F}_{q^b} is a subfield of $\mathbb{F}_{q^{b'}}$.

We use induction on i to show that there exists positive integer b_i such that the element $\sum_{x \in X_{i,q^{b_i}}} x\eta$ is in V . For $i = 1$, this is true by Lemma 2.6. Now assume that $\sum_{x \in X_{i,q^{b_i}}} x\eta$ is in V , we show that $\sum_{x \in X_{i+1,q^{b_{i+1}}}} x\eta$ is in V for some b_{i+1} .

Let $c_1, \dots, c_{q^{b_i}+1}$ be a complete set of representatives of all cosets of $\mathbb{F}_{q^{b_i}}^*$ in $\mathbb{F}_{q^{2b_i}}^*$. Choose $t_1, \dots, t_{q^{b_i}+1} \in T$ such that $\beta_i(t_j) = c_j$ for $j = 1, \dots, q^{b_i} + 1$. Note that $t^{-1}\eta = \eta$ for any $t \in T$. Thus

$$\sum_{j=1}^{q^{b_i}+1} t_j \sum_{x \in U_{\beta_i,q^{b_i}}} x\eta = q^{b_i}\eta + \sum_{x \in U_{\beta_i,q^{2b_i}}} x\eta.$$

Since $X_{i,q^{b_i}} = X_{i+1,q^{b_i}}U_{\beta_i,q^{b_i}}$ and $\sum_{x \in X_{i,q^{b_i}}} x\eta$ is in V , we see

$$\begin{aligned} \xi &= \sum_{j=1}^{q^{b_i}+1} t_j \sum_{x \in X_{i,q^{b_i}}} x\eta \\ &= \sum_{j=1}^{q^{b_i}+1} t_j \sum_{y \in X_{i+1,q^{b_i}}} y \sum_{x \in U_{\beta_i,q^{b_i}}} x\eta \\ &= \sum_{y \in X_{i+1,q^{b_i}}} \sum_{j=1}^{q^{b_i}+1} t_j y t_j^{-1} (t_j \sum_{x \in U_{\beta_i,q^{b_i}}} x\eta) \in V. \end{aligned}$$

Choose b_{i+1} such that all $\alpha_m(t_j)$ ($r \geq m \geq i$) are contained in $\mathbb{F}_{q^{b_{i+1}}}$. Then $\mathbb{F}_{q^{b_{i+1}}}$ contains $\mathbb{F}_{q^{2b_i}}$. Thus $t_j y t_j^{-1}$ is in $X_{i+1, q^{b_{i+1}}}$ for any $y \in X_{i+1, q^{b_i}}$. Let $Z \in kG$ be the sum of all elements in $X_{i+1, q^{b_{i+1}}}$. Then we have

$$(1) \quad Z\xi = q^{(r-i)b_i} Z \sum_{j=1}^{q^{b_i}+1} t_j \sum_{x \in U_{\beta_i, q^{b_i}}} x\eta = q^{(r-i)b_i} Z(q^{b_i}\eta + \sum_{x \in U_{\beta_i, q^{2b_i}}} x\eta) \in V.$$

Since $\sum_{x \in X_{i, q^{b_i}}} x\eta$ is in V , we have $\sum_{x \in X_{i, q^{2b_i}}} x\eta \in V$. Thus

$$(2) \quad Z \sum_{x \in X_{i, q^{2b_i}}} x\eta = q^{2(r-i)b_i} Z \sum_{x \in U_{\beta_i, q^{2b_i}}} x\eta \in V.$$

Since $q \neq 0$ in k , combining (1) and (2) we see that $Z\eta \in V$, i.e., $\sum_{x \in X_{i+1, q^{b_{i+1}}}} x\eta$ is in V .

Note that $X_{r, q^{b_r}} = U_{\beta_r, q^{b_r}}$. Now we have

$$\sum_{x \in U_{r, q^{b_r}}} x\eta \in V \quad \text{and} \quad \sum_{x \in U_{r, q^{2b_r}}} x\eta \in V.$$

The above arguments show that

$$q^{b_r}\eta + \sum_{x \in U_{r, q^{2b_r}}} x\eta \in V.$$

Therefore $q^{b_r}\eta$ is in V . So V contains $kG\eta = \text{St}$, hence $V = \text{St}$. The theorem is proved.

2.8. Remark. Let $\text{St}_a = kG_{q^a}\eta$. Then St_a is the Steinberg module of kG_{q^a} , which is not irreducible in general. As an example, say, $G = SL_2(\mathbb{F}_q)$ and q is odd, $\text{char } k = 2$. Since $q^a + 1$ is always divisible by $2 = \text{char } k$, St_a is not irreducible for any positive number a (see [S, Theorems 3]). However, by Theorem 2.2, St is irreducible kG -module.

3. Non-quasi-finite irreducibility of certain Steinberg modules

In this section we show that for certain special linear groups the Steinberg modules of the groups are not quasi-finite with respect to some natural quasi-finite sequences of the groups, see Proposition 3.2.

3.1. By definition, a group G is quasi-finite if G has a sequence $G_1, G_2, \dots, G_n, \dots$ of finite subgroups such that G is the union of all G_i and for any positive integers i, j there exists integer r such that G_i and G_j are contained in G_r . The sequence G_1, G_2, G_3, \dots is called a quasi-finite sequence of G . An irreducible module (or representation) M of G is *quasi-finite* (with respect to the quasi-finite sequence G_1, G_2, G_3, \dots) if it has a sequence of subspaces M_1, M_2, M_3, \dots of M such that (1) each M_i is an irreducible G_i -submodule of M , (2) if G_i is a subgroup of G_j , then M_i is a subspace of M_j , (3) M is the union of all M_i . The sequence M_1, M_2, M_3, \dots will be called a quasi-finite sequence of M . See [X, 1.8]

The following question was raised in [4, 1.8]: is every irreducible G -module quasi-finite (with respect to a certain quasi-finite sequence of G).

The main result of this section is the following result.

3.2. Proposition. Let $G = SL_n(\bar{\mathbb{F}}_q)$ and k a field of positive characteristic. Assume that $\text{char } k$ divides $(1 + q^a)(1 + q^a + q^{2a}) \cdots (1 + q^a + \cdots + q^{(n-1)a})$ for all positive integers a . If a quasi-finite sequence of G is a subsequence of $SL_n(\mathbb{F}_q), SL_n(\mathbb{F}_{q^2}), SL_n(\mathbb{F}_{q^3}), SL_n(\mathbb{F}_{q^4}), \dots$, then the Steinberg module St of kG is not quasi-finite with respect to the quasi-finite sequence.

Proof. Let $G_1, G_2, \dots, G_n, \dots$ be a quasi-finite sequence of G . Assume that the quasi-finite sequence is a subsequence of $SL_n(\mathbb{F}_q), SL_n(\mathbb{F}_{q^2}), SL_n(\mathbb{F}_{q^3}), SL_n(\mathbb{F}_{q^4}), \dots$. If St is quasi-finite with respect to this quasi-finite sequence, then there exists a sequence of subspaces M_1, M_2, M_3, \dots of St such that (1) each M_i is an irreducible G_i -submodule of M , (2) if G_i is a subgroup of G_j , then M_i is a subspace of M_j , (3) M is the union of all M_i .

Choose a nonzero element $v \in M_1$. By the proof of Theorem 2.2, there exists $x \in kG$ such that $xv = \eta$. Since G is the union of all $H_a = SL_n(\mathbb{F}_{q^a})$, there exists positive integer i such that $x \in kH_i$. Since G is the union of all G_a , there exists j such that H_i is included in G_j . Note that $G_j = H_{j'}$ for some positive integer j' . Choose integer d such that G_d includes both G_1 and G_j . Then M_d includes M_1 and

M_j . Moreover, $x \in kG_d$, so that $xv = \eta$ is in M_d and M_d includes $kG_d\eta$. Since $G_d = H_{d'}$ for some d' and $kH_{d'}\eta$ is not irreducible, M_d is not irreducible. This contradicts the assumption that M_d is irreducible G_d -module. The proposition is proved.

3.3. Remark. Assume that n is power of a prime p' and k has characteristic p' . If n, q are coprime, then n divides $(1 + q^a)(1 + q^a + q^{2a}) \cdots (1 + q^a + \cdots + q^{(n-1)a})$ for all positive integers a . To see this, let $A_m = 1 + q^a + \cdots + q^{(m-1)a}$, $m = 2, \dots, n$. Since n, q are coprime, if $A_m \equiv 1 \pmod{n}$, then $m > 2$ and A_{m-1} is divisible by n . If $A_m \not\equiv 1 \pmod{n}$ for $m = 2, \dots, n$, then either some A_m is divisible by n or $A_m \equiv A_l \pmod{n}$ for some $n \geq m > l \geq 2$. Since n, q are coprime, we have $m \geq l + 2$. Then A_{m-l} is divisible by n . By Proposition 2.2, in this case, the Steinberg module St of $kSL_n(\bar{\mathbb{F}}_q)$ is not quasi-finite for a quasi-finite sequence of G whenever it is a subsequence of $SL_n(\mathbb{F}_q)$, $SL_n(\mathbb{F}_{q^2})$, $SL_n(\mathbb{F}_{q^3})$, $SL_n(\mathbb{F}_{q^4})$,

However, it is not clear that whether St is quasi-finite with other quasi-finite sequences of $SL_n(\bar{\mathbb{F}}_q)$.

For other reductive groups, one discusses similarly.

3.4. Assume that G is quasi-finite and has sequence of normal subgroups $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that all G_i/G_{i-1} are abelian. Xi asks whether any irreducible $\mathbb{C}G$ -module is isomorphic to the induced module of a one dimensional module of a subgroup of G (see [X, 1.12]). The question has a negative answer ever for finite groups, for instance, the two-dimensional irreducible complex representation of $SL_2(\mathbb{F}_3)$ is an counterexample. Perhaps for the question the condition of all G_i/G_{i-1} being abelian should be further strengthened to all G_i/G_{i-1} being cyclic.

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